

PLANAR LEFSCHETZ FIBRATIONS AND STEIN STRUCTURES WITH DISTINCT OZSVÁTH-SZABÓ INVARIANTS ON CORKS

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ABSTRACT. Thanks to a result of Lisca and Matić and a refinement by Plamenevskaya, it is known that on a 4-manifold with boundary Stein structures with non-isomorphic Spin^c structures induce contact structures with distinct Ozsváth-Szabó invariants. Here we give an infinite family of examples showing that converse of Lisca-Matić-Plamenevskaya theorem does not hold in general. Our examples arise from Mazur type corks.

1. INTRODUCTION

For any contact structure ξ on a 3-manifold Y , let $c^+(\xi) \in HF^+(-Y)$ denote its Ozsváth-Szabó invariant. Recall Lisca-Matić-Plamenevskaya theorem:

Theorem 1.1. [LM97, Theorem 1.2] [Pla04, Theorem 2] *Let W be a smooth compact 4-manifold with boundary W equipped with two Stein structures J_1 and J_2 with associated Spin^c structures \mathfrak{s}_1 and \mathfrak{s}_2 on W , and the induced contact structures ξ_1 and ξ_2 on ∂W . If \mathfrak{s}_1 and \mathfrak{s}_2 are not isomorphic then ξ_1 and ξ_2 are not isotopic; In fact $c^+(\xi_1) \neq c^+(\xi_2)$*

In the light of the above theorem a natural question to ask is whether the Spin^c structure of a Stein filling completely determines the Ozsváth-Szabó invariant of the induced contact structure. An evidence towards a positive answer was provided in a work of Karakurt [Kar14, Proposition 1.2] where it was shown that the Ozsváth-Szabó invariant depends only on the first Chern class of the Stein filling on W when the total space of the filling is a special type of plumbing. Our main result suggests that the answer is in general negative. To state it let $\pi : HF^+(-\partial Y) \rightarrow HF^{\text{red}}(-\partial Y)$ be the natural projection map from the plus flavor to reduced Heegaard Floer homology.

Theorem 1.2. *There exists an infinite family $\{W^n : n \in \mathbb{N}\}$ of compact contractible 4-manifolds with boundary and Stein structures J_1^n and J_2^n on W^n satisfying the following properties:*

- (1) *The Spin^c structures \mathfrak{s}_1^n and \mathfrak{s}_2^n associated to J_1^n and J_2^n , respectively, are the same for every $n \in \mathbb{N}$.*
- (2) *The induced contact structures ξ_1^n and ξ_2^n on ∂W^n have distinct Ozsváth-Szabó invariants, in fact $\pi(c^+(\xi_1^n)) \neq 0$ and $\pi(c^+(\xi_2^n)) = 0$, for every $n \in \mathbb{N}$.*
- (3) *the Casson invariant of ∂W^n is given by $\lambda(\partial W^n) = 2n$ for every $n \in \mathbb{N}$.*
- (4) *∂W^n is irreducible for every $n \in \mathbb{N}$.*

Our examples W^n are Mazur type manifolds obtained from the symmetric link L^n in Figure 1 by putting a dot on one of the components and attaching a 0-framed 2-handle to the other component as in Figure 2. Note that the manifold W^1 is the Akbulut cork. A Stein structure J_1^n on W^n can immediately be obtained by drawing a Legendrian representative of the attaching circle of the 2-handle and stabilizing as necessary to make the framing one less than the Thurston-Bennequin

framing. Even though the choice of stabilizations is not unique, and different stabilizations potentially yield Stein structures with distinct Ozsváth-Szabó invariants, the direct computation of these invariants does not seem plausible. Hence we take a different approach and construct the second Stein structure J_2^n using the Loi-Piergallini-Akbulut-Ozbagci correspondence between Stein structures and positive allowable Lefschetz fibrations (PALFs in short). Our key observation is that W^n admits a planar PALF, that is, a PALF with planar fiber. This was already shown for the Akbulut cork W^1 by Ukida [Uki]. The main result for W^1 then immediately follows by bringing together some known facts in the literature (see our proof below). One can easily promote Ukida's example to an infinite family by repeatedly taking boundary sums of W^1 . The irreducibility of ∂W^n shows that our examples do not arise in this manner. In the body of our work we generalize Ukida's planar PALF construction to W^n , and compute the Casson invariants to distinguish ∂W^n 's. Along the way we also prove that ∂W^n is obtained from S^3 by $1/n$ -surgery on a knot, a fact we find interesting in its own right.

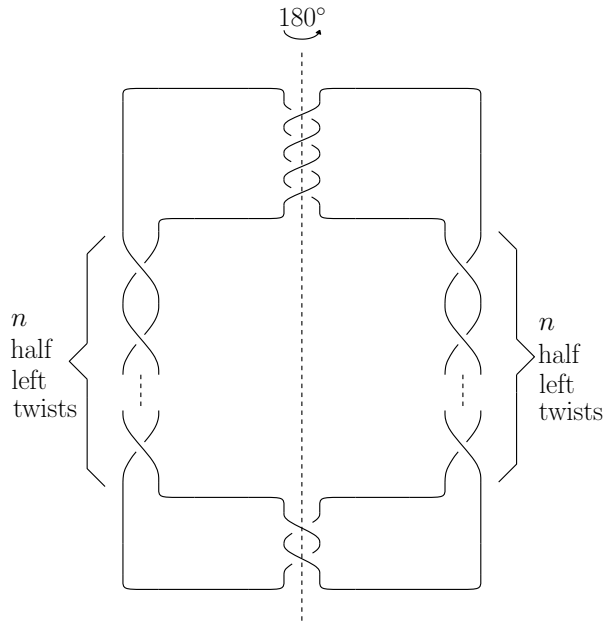


FIGURE 1. Symmetric picture of L^n . The indicated involution exchanges the components.

2. PROOF OF THE MAIN THEOREM

First recall the terminology from [AK12]. Let L be a link in S^3 with two components $K_1 \cup K_2$. We say that L is an *admissible* link if it satisfies the following conditions:

- (1) Both K_1 and K_2 are unknotted.
- (2) An involution of S^3 exchanges K_1 and K_2 .
- (3) The linking number of K_1 and K_2 is ± 1 .
- (4) Carve out a disk bounded by K_1 and regard $K_2 \subset S^1 \times S^2 = \partial(S^1 \times B^3)$ equipped with the unique Stein fillable contact structure. Then the maximal Thurston-Bennequin number of K_2 is at least $+1$.

0

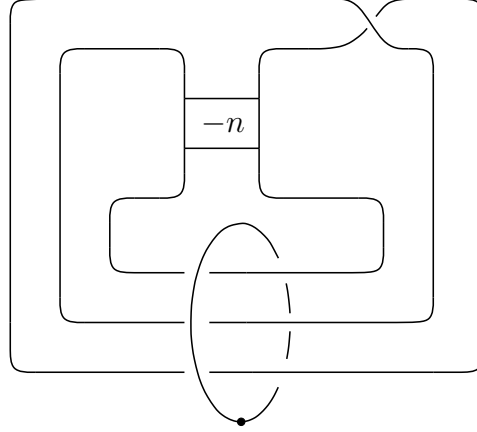


FIGURE 2. The handlebody $W^n = W(L^n)$. The box indicates n full left twists.

From an admissible link, we can construct an obvious contractible Stein handlebody $W(L)$ by putting a dot on K_1 , and attaching a 2-handle along some Legendrian representative K_2 with framing one less than the Thurston-Bennequin framing (this is possible thanks to the last condition).

As in the introduction, let L^n be the link given in Figure 1, and let $W^n := W(L^n)$ denote the corresponding handlebody obtained by putting a dot on one of the components and 0 on the other one as in Figure 2.

Proposition 2.1. *For every $n \in \mathbb{N}$, the link L^n is admissible.*

Proof. In Figure 1 the 180° rotation about the dashed axis exchanges the components of L^n . It is also clear from the figure that both components of L^n are unknotted and the linking number of these components is ± 1 . We must check that the handlebody W^n is Stein. By Eliashberg's characterization, it suffices to show that the attaching circle of the 2-handle has maximal Thurston Bennequin number $TB \geq 1$ in $S^1 \times S^2$. In Figure 3, we draw a Legendrian representative of the attaching circle of the 2-handle on $S^1 \times S^2$. From the figure we see that the writhe is $2n + 1$ and half the number of cusps is $2n - 1$, implying that $TB \geq 2$. Hence a stabilization of the figure gives a Stein handlebody picture of W^n .

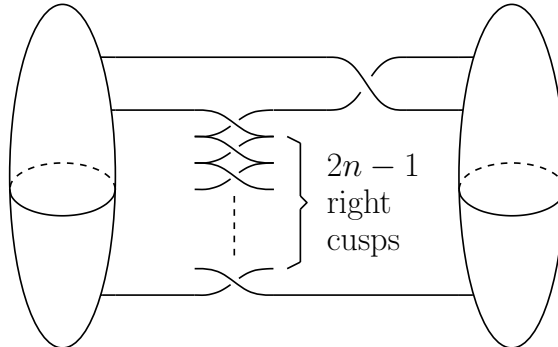


FIGURE 3. Cork W^n as a Stein handlebody (Need to stabilize the 2-handle once).

□

Denote the corresponding Stein structure on W^n (for any choice of stabilization) by J_1^n , and the induced contact structure on ∂W^n by ξ_1^n . The following result shows that $\pi(c^+(\xi_1^n)) \neq 0$.

Theorem 2.2. [AK12, Theorem 4.1] *Let L be an admissible link, $W(L)$ be the corresponding Stein handlebody and ξ the induced contact structure on $\partial W(L)$. Then $\pi(c^+(\xi)) \neq 0$.*

It is important for the above theorem that the Stein structure is the one coming from the handlebody picture associated to an admissible link.

Proposition 2.3. *The manifold W^n admits a planar PALF for every $n \in \mathbb{N}$.*

Proof. For $n = 1$, this result was proved by Ukida in [Uki]. We generalize Ukida's argument in an obvious manner. We apply the handlebody moves indicated in Figure 4. Clearly the last diagram gives the total space of PALF whose fibers are disks with $n+3$ holes and monodromy is the following product of right handed Dehn twists $t_a t_b t_c t_{d_1} \cdots t_{d_n}$ where $a, b, c, d_1 \dots d_n$ are the curves indicated in Figure 5. \square

Now that we know W^n admits a planar PALF, by results of Loi-Piergallini [LP01] and Akbulut-Ozbagci [AO01] there is a corresponding Stein structure on W^n which we denote by J_2^n . Let ξ_2^n be the induced contact structure on ∂W^n . Note that ξ_2^n is supported by a planar open book. The next result which is due to Ozsváth-Stipsicz-Szabó implies that $\pi(c^+(\xi_2^n)) = 0$

Theorem 2.4. [OSS05, Theorem 1.2] *Let Y be a 3-manifold and ξ a contact structure on Y . Suppose that ξ is supported by a planar open book decomposition. Then $\pi(c^+(\xi)) = 0$.*

We have just observed that the Ozsváth-Szabó invariants of ξ_1^n and ξ_2^n satisfy the required properties. It is clear that the induced Spin^c structures \mathfrak{s}_1^n and \mathfrak{s}_2^n are the same since W^n is contractible. To prove the rest of the theorem first we observe that the boundary of each W^n is the manifold $S^3_{1/n}(K)$ which is obtained from S^3 by $1/n$ -surgery on the knot K on the left hand side of Figure 6.

Lemma 2.5. *We have $\partial W^n = S^3_{1/n}(K)$ for all $n \in \mathbb{N}$.*

Proof. This was proved for $n = 1$ by Akbulut and Kirby [AK79, Proposition 1-(3)]. One can easily modify their argument to see the proof in the general case. Alternatively we can apply the handlebody moves depicted in Figure 7 and Figure 8 to show that ∂W^n is obtained from S^3 by $1/n$ -surgery on a knot. It is easy to see that the knots in Figure 6 and at the end of Figure 8 are isotopic. \square

Now the irreducibility of ∂W^n follows from a result of Gordon and Luecke [GL89] which says that if a reducible manifold appears as a surgery on a knot in S^3 then one of its summands must be a lens space. Since ∂W^n is an integral homology sphere, it cannot have any non-trivial lens space summands.

Lemma 2.6. *The Alexander polynomial of the knot K is given by $\Delta_K(t) = 2t^2 - 5t + 2$.*

Proof. We use the Seifert surface of K that is indicated on the right hand side of Figure 6. With respect to the homology generators α, β , the Seifert matrix is given by

$$S = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}.$$

Then the Alexander polynomial is $\Delta_K(t) = \text{Det}(S - tS^T) = 2t^2 - 5t + 2$. \square

From the above Lemma we conclude that the Casson invariant of ∂W^n is given by

$$\lambda(\partial W^n) = \frac{n}{2} \Delta_K''(1) = 2n,$$

which finishes the proof of Theorem 1.2.

3. FINAL REMARKS

We anticipate that there are lots of symmetric links that give rise to contractible manifolds with distinct Stein structures. For example in [Oba15], Oba constructed an infinite family of planar PALFs on Mazur type contractible manifolds. We do not know whether the links defining Oba's manifold are admissible. If they are, then they could be used in the proof instead of L^n .

Finally, our techniques can only distinguish those Stein structures supported by planar PALFs from those which cannot, so we are unable to detect more than 3 distinct Stein structures on a contractible manifold. Hence the following is still an interesting open problem

Question 3.1. Given $k \geq 3$, is there a contractible manifold 4-manifold (with irreducible boundary) which admits k distinct Stein structures?

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REFERENCES

- [AK79] Selman Akbulut and Robion Kirby, *Mazur manifolds*, Michigan Math. J. **26** (1979), no. 3, 259–284. MR 544597
- [AK12] Selman Akbulut and Çağrı Karakurt, *Action of the cork twist on Floer homology*, Proceedings of the Gökova Geometry-Topology Conference 2011, Int. Press, Somerville, MA, 2012, pp. 42–52. MR 3076042
- [AO01] Selman Akbulut and Burak Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001), 319–334 (electronic). MR 1825664
- [GL89] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), no. 2, 371–415. MR 965210
- [Kar14] Çağrı Karakurt, *Contact structures on plumbed 3-manifolds*, Kyoto J. Math. **54** (2014), no. 2, 271–294. MR 3215568
- [LM97] P. Lisca and G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. **129** (1997), no. 3, 509–525. MR 1465333
- [LP01] Andrea Loi and Riccardo Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4* , Invent. Math. **143** (2001), no. 2, 325–348. MR 1835390
- [Oba15] Takahiro Oba, *A note on Mazur type Stein fillings of planar contact manifolds*, Topology Appl. **193** (2015), 302–308. MR 3385100
- [OSS05] Peter Ozsváth, András Stipsicz, and Zoltán Szabó, *Planar open books and Floer homology*, Int. Math. Res. Not. (2005), no. 54, 3385–3401. MR 2200085 (2006j:57056)
- [Pla04] Olga Plamenevskaya, *Contact structures with distinct Heegaard Floer invariants*, Math. Res. Lett. **11** (2004), no. 4, 547–561. MR 2092907
- [Uki] Takuya Uki, *A genus zero Lefschetz fibration on the Akbulut cork*, preprint 2014, arXiv:1406.5865.

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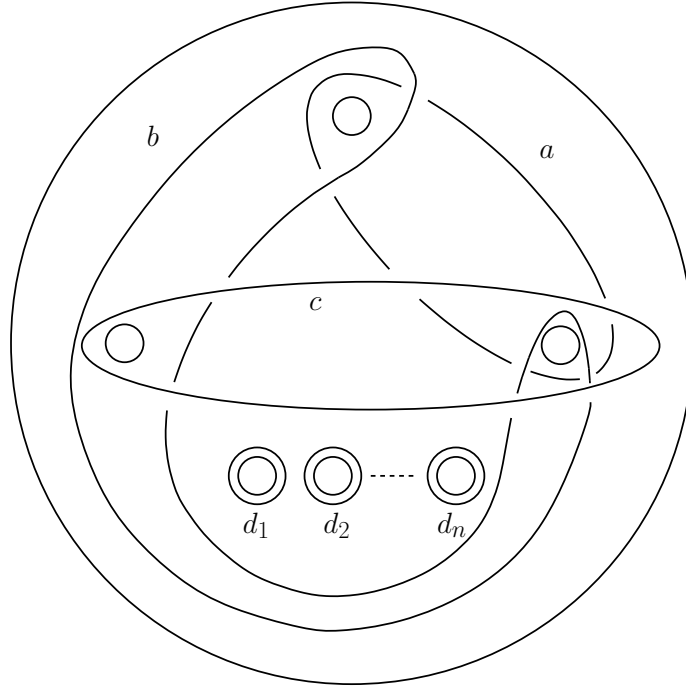
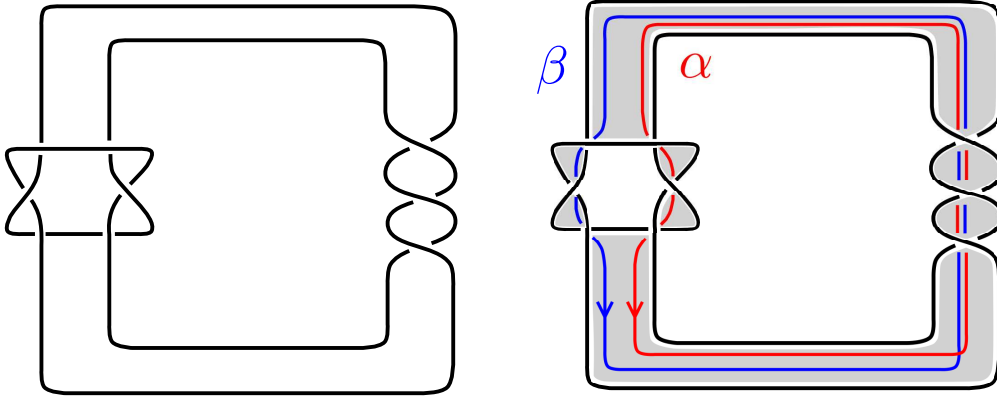
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FIGURE 5. Our planar PALF on W^n FIGURE 6. The knot K is on the left. The Seifert surface of K together with its homology generators are on the right.

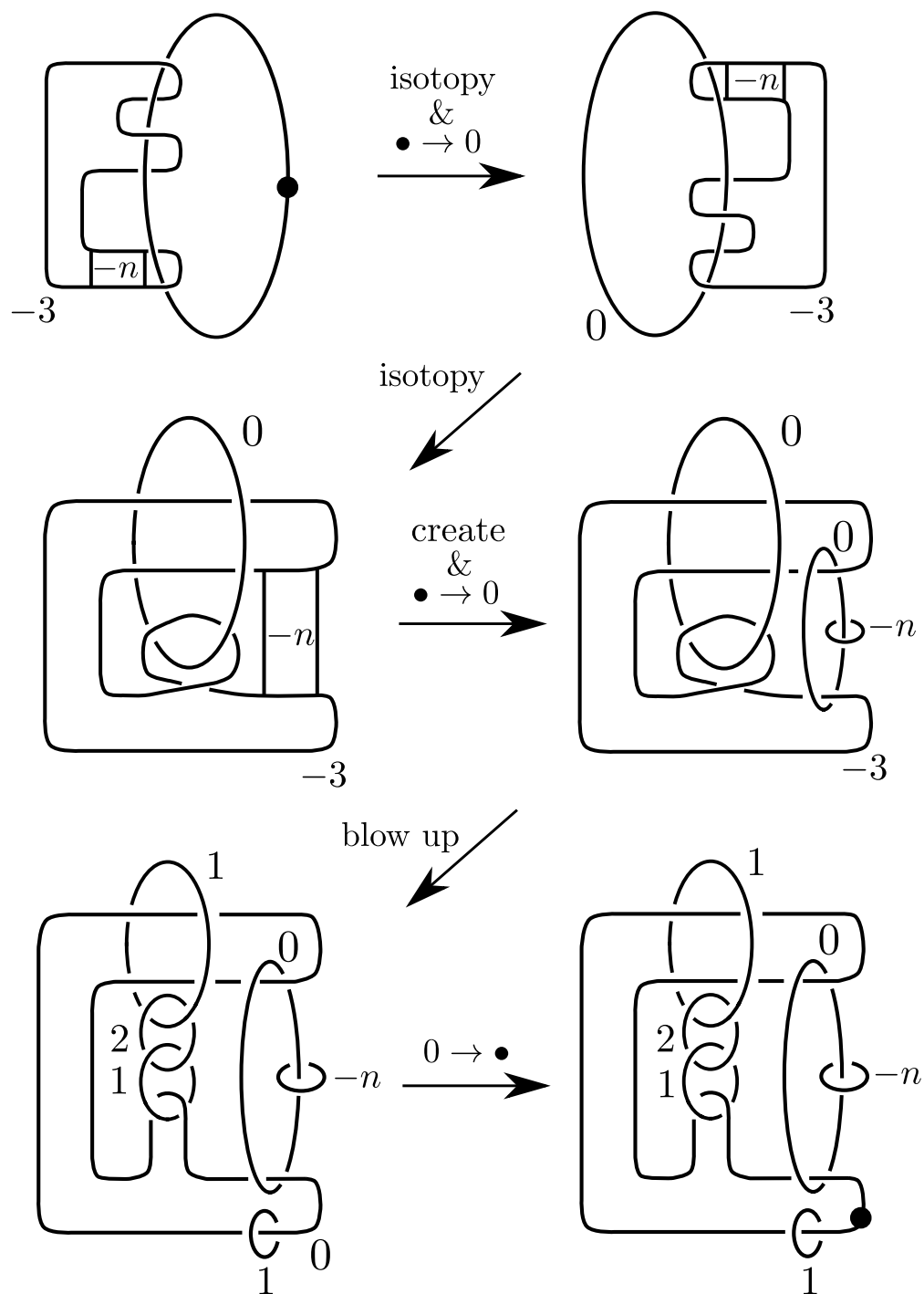


FIGURE 7.

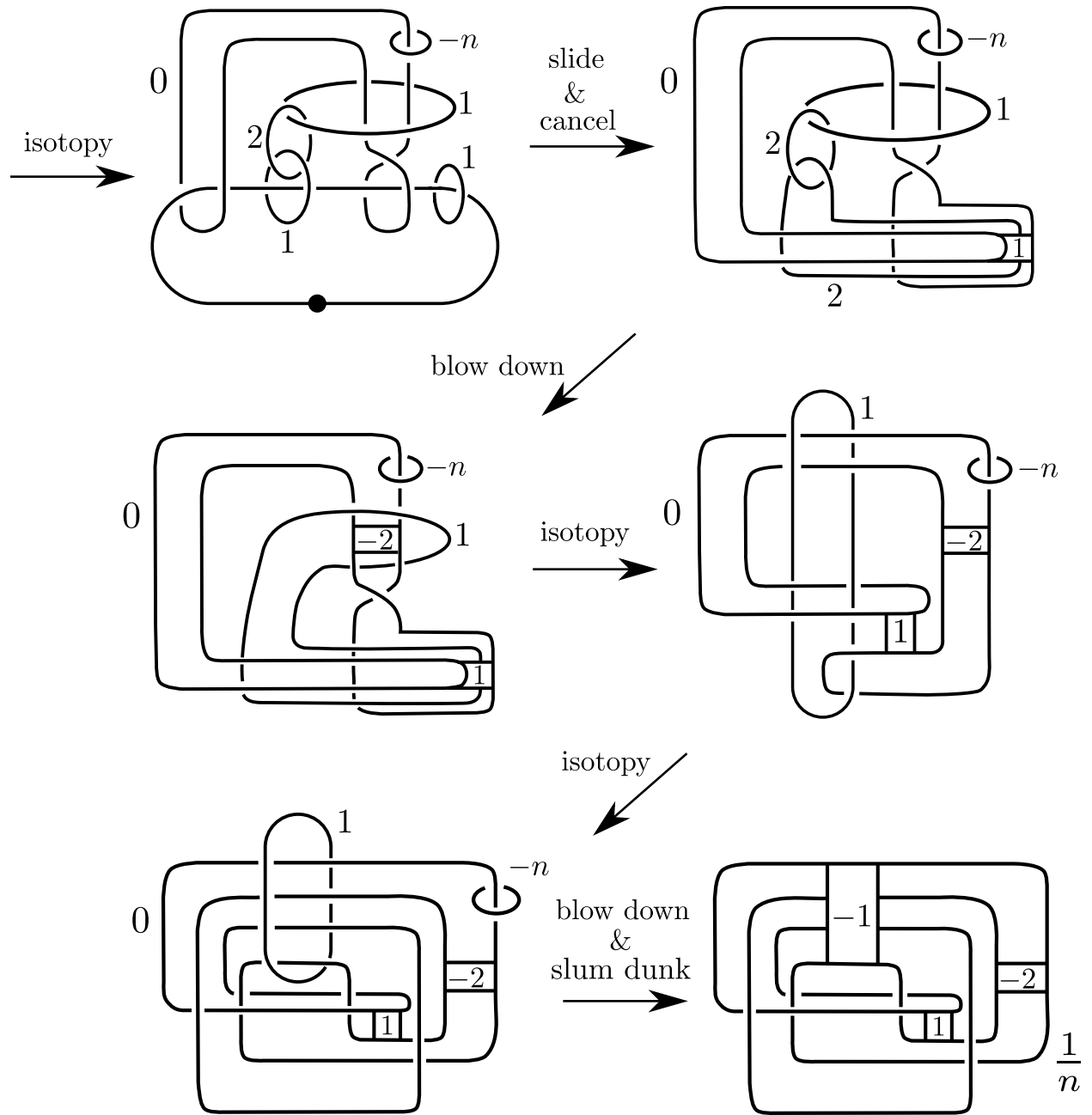


FIGURE 8.